

# The martingale representation in a progressive enlargement of a filtration with jumps \*

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## Abstract

In this paper, we assume that the filtration  $\mathbb{F}$  is generated by a  $d$ -dimensional Brownian motion  $W = (W_1, \dots, W_d)'$  as well as an integer-valued random measure  $\mu(du, dy)$ . The random variable  $\tilde{\tau}$  is the default time and  $L$  is the default loss. Let  $\mathbb{G} = \{\mathcal{G}_t; t \geq 0\}$  be the progressive enlargement of  $\mathbb{F}$  by  $(\tilde{\tau}, L)$ , i.e.  $\mathbb{G}$  is the smallest filtration including  $\mathbb{F}$  such that  $\tilde{\tau}$  is a  $\mathbb{G}$ -stopping time and  $L$  is  $\mathcal{G}_{\tilde{\tau}}$ -measurable. Under the density hypothesis, we consider the  $\mathbb{G}$ -decomposition of a  $(P, \mathbb{F})$  martingale and the representation of a  $\mathbb{G}$ -martingale. We characterize the conditional density process by  $p_s(s, l)$ ,  $\theta_1(u; s, l)\mathbb{I}_{u>s}$  and  $\theta_2(u, y; s, l)\mathbb{I}_{u>s}$ , which allows us to describe the survival process  $G$  explicitly. Then we give the explicit  $\mathbb{G}$ -decomposition of a  $\mathbb{F}$  martingale and obtain the predictable representation theorems both for a  $(P, \mathbb{G})$ -martingale and a  $(P^*, \mathbb{G})$ -martingale, which are different as shown in Callegaro, Jeanblanc and Zargari(2010) <sup>[5]</sup>.

**Key words:** default time and default loss, progressive enlargement of filtration, conditional density, canonical decomposition, martingale representation.

## 1 Introduction

In this paper, we assume that the filtration  $\mathbb{F}$  is generated by a  $d$ -dimensional Brownian motion  $W = (W_1, \dots, W_d)'$  as well as an integer-valued random measure  $\mu(du, dy)$ . Let  $\tilde{\tau} \geq 0$  be the default time and  $L$  be the default loss, which are random variables. Let  $\mathbb{G} = \{\mathcal{G}_t; t \geq 0\}$  be the progressive enlargement of  $\mathbb{F}$  by  $(\tilde{\tau}, L)$ , i.e.  $\mathbb{G}$  is the smallest filtration including  $\mathbb{F}$  such that  $\tilde{\tau}$  is a  $\mathbb{G}$ -stopping time and  $L$  is  $\mathcal{G}_{\tilde{\tau}}$ -measurable. We note here that the progressive enlargement filtration in this paper is different from traditional progressive enlargement of filtration in the literature, see e.g. in Jeulin(1980)<sup>[15]</sup>, Jacod(1987)<sup>[10]</sup>, Jeanblanc, Yor and Chesney(2009)<sup>[11]</sup>, El Karoui, Jeanblanc and Jiao(2009)<sup>[8]</sup>, Jeanblanc and Le Cam(2009)<sup>[12]</sup>, Jeanblanc and Song(2010)<sup>[13]</sup>, Callegaro, Jeanblanc and Zargari(2010)<sup>[5]</sup>

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and Jeanblanc and Song(2012)<sup>[14]</sup>, among many others. In traditional progressive enlargement, the authors didn't consider the more practical situation that once the default happens, the default loss is immediately generated. Hence the filtration  $\mathbb{G}$  we consider here including the default time and the default loss simultaneously is more interesting and realistic.

It is well known that for a general enlargement of filtration, a  $(P, \mathbb{F})$ -martingale might not be a  $(P, \mathbb{G})$ -semimartingale, thus Jacod(1987)<sup>[10]</sup> and El Karoui, Jeanblanc and Jiao(2009)<sup>[8]</sup> proposed the concept of density hypothesis (H'), which is adapted in the following:

**Assumption 1.1.** *i) the  $\mathbb{F}$ -regular conditional law of  $(\tilde{\tau}, L)$  is equivalent to the law of  $(\tilde{\tau}, L)$ , i.e*

$$P(\tilde{\tau} \in ds, L \in dl) \sim \eta(ds, dl), \text{ for every } t \geq 0;$$

*ii)  $\eta(ds, dl)$  has no atoms.*

From Assumption 1.1, one can see that there exists a so-called **conditional density**  $p_t(s, l)$  to describe  $\mathbb{F}$ -regular conditional law of  $(\tilde{\tau}, L)$ , such that

$$P((\tilde{\tau}, L) \in B | \mathcal{F}_t) = \int \int_B p_t(s, l) \eta(ds, dl), \text{ for every } t \geq 0, P\text{-a.s.}$$

One can see that  $\{p_t(s, l); t \geq 0\}$  is a  $(P, \mathbb{F})$ -martingale. In this paper, we investigate  $\{p_t(s, l); t \geq 0\}$  more deeply and find that under Assumption 2.1, it is completely determined by  $p_s(s, l)$ ,  $\theta_1(u; s, l)' \mathbb{I}_{u>s}$  and  $\theta_2(u, y; s, l) \mathbb{I}_{u>s}$  as

$$\begin{aligned} p_t(s, l) &= E[p_s(s, l) | \mathcal{F}_t] + p_s(s, l) \mathbb{I}_{t \geq s} + \int_0^t p_{u-}(s, l) \theta_1(u; s, l)' \mathbb{I}_{u>s} dW(u) \\ &\quad + \int_0^t \int_E p_{u-}(s, l) \theta_2(u, y; s, l) \mathbb{I}_{u>\tilde{\tau}} \{\mu(du, dy) - \nu(du, dy)\}. \end{aligned}$$

We show in Theorem 2.9 that  $p_s(s, l)$ ,  $\theta_1(u; s, l)' \mathbb{I}_{u>s}$  and  $\theta_2(u, y; s, l) \mathbb{I}_{u>s}$  must satisfy the following condition

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}} p_s(s, l) \eta(ds, dl) &= 1 - \int_0^\infty \left\{ \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_1(u; s, l) \eta(ds, dl) \right\}' dW(u) \\ &\quad - \int_0^\infty \int_E \left\{ \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_2(u, y; s, l) \eta(ds, dl) \right\} \{\mu(du, dy) - \nu(du, dy)\}, \end{aligned} \quad (1.1)$$

where

$$\begin{aligned} Z_{s,l}^{\theta_1, \theta_2}(t) &= \exp \left\{ \int_s^t \theta_1(u; s, l)' dW(u) - \frac{1}{2} \int_s^t \|\theta_1(u; s, l)\|^2 du \right\} \\ &\quad \times \exp \left\{ \int_s^t \int_E \{ \ln(1 + \theta_2(u, y; s, l)) \} \mu(du, dy) - \int_s^t \int_E \theta_2(u, y; s, l) \nu(du, dy) \right\}. \end{aligned}$$

Then we prove that the corresponding survival process  $G_t$  has the following Doob-Meyer's decomposition

$$\begin{aligned} G_t &= 1 - \int_0^t \int_{\mathbb{R}} p_s(s, l) \eta(ds, dl) \\ &\quad - \int_0^t \left\{ \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_1(u; s, l) \eta(ds, dl) \right\}' dW(u) \\ &\quad - \int_0^t \int_E \left\{ \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_2(u, y; s, l) \eta(ds, dl) \right\} \{\mu(du, dy) - \nu(du, dy)\}, \end{aligned} \quad (1.2)$$

which is completely determined by  $p_s(s, l)$ ,  $\theta_1(u; s, l)\mathbb{I}_{u>s}$  and  $\theta_2(u, y; s, l)\mathbb{I}_{u>s}$ , which appears to be quite interesting.

Similar to Callegaro, Jeanblanc and Zargari(2010)<sup>[5]</sup>, we show that  $M_t := M_1(t)\mathbb{I}_{t<\tilde{\tau}} + M_2(t; \tilde{\tau}, L)\mathbb{I}_{t\geq\tilde{\tau}}$  is a  $(P, \mathbb{G})$ -martingale if only if the following two conditions are satisfied:

- (i) for  $\eta$ -almost every  $u \geq 0$  and  $l \in \mathbb{R}$ ,  $\{\hat{y}_t(u; l)p_t(u; l); t \geq u\}$  is a  $(P, \mathbb{F})$ -martingale;
- (ii) the process  $\{\tilde{y}_t G_t + \int_0^t \int_{\mathbb{R}} \hat{y}_u(u, l)p_u(u, l)\eta(du, dl); t \geq 0\}$  is a  $(P, \mathbb{F})$ -martingale.

Then we mainly study the  $\mathbb{G}$ -decomposition of a  $(P, \mathbb{F})$  martingale and the representation of a  $\mathbb{G}$ -martingale. This representation is very important also as it has many applications in mathematical finance, see Duffie and Huang(1986)<sup>[6]</sup>, Karatzas and Pikovsky(1996)<sup>[18]</sup>, Amendinger, Becherer and Schweizer(2003)<sup>[2]</sup>, Ankirchner, Dereichner and Imkeller(2005)<sup>[3]</sup>, Jiao and Pham(2009)<sup>[16]</sup> and Eyraud-Loisel(2010)<sup>[7]</sup>, etc. Let  $m$  be a càdlàg  $(P, \mathbb{F})$ -local martingale of the following form

$$m_t = m_0 + \int_0^t \xi_1(u)' dW(u) + \int_0^t \int_E \xi_2(u, y) \{\mu(du, dy) - \nu(du, dy)\},$$

we show that

$$\begin{aligned} X_t = m_t + \int_0^{t \wedge \tilde{\tau}} \frac{1}{G_{u-}} \left\{ \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \left\{ \xi_1(u)' \theta_1(u; s, l) \right. \right. \\ \left. \left. + \int_E \theta_2(u, y; s, l) \xi_2(u, y) F_u(dy) \right\} \eta(ds, dl) \right\} du \\ - \int_{\tilde{\tau}}^t \left\{ \xi_1(u)' \theta_1(u; \tilde{\tau}, L) + \int_E \theta_2(u, y; \tilde{\tau}, L) \xi_2(u, y) F_u(dy) \right\} du \end{aligned}$$

is a  $(P, \mathbb{G})$ -local martingale, from which one can see the  $\mathbb{G}$ -decomposition of  $m$ . It is notable that the derived  $\mathbb{G}$ -decomposition also only depends on  $p_s(s, l)$ ,  $\theta_1(u; s, l)\mathbb{I}_{u>s}$  and  $\theta_2(u, y; s, l)\mathbb{I}_{u>s}$ . In the end, we obtain the predictable representation theorems both for a  $(P, \mathbb{G})$ -martingale and a  $(P^*, \mathbb{G})$ -martingale. Although recently Jeanblanc and Song(2012)<sup>[14]</sup> gave the more general discussion for martingale representation property in traditional progressively enlarged filtration, it does not apply to our filtration  $\mathbb{G}$  which is enlarged by the default time and the default loss. Furthermore, since we have the density hypothesis, our method is more simple and more directly, and the results are also seen to be different.

The paper is organized as follows: In Section 2, we characterize the conditional density process and obtain a more explicit form of the Doob-Meyer's decomposition of the survival process. Then we provide a necessary and sufficient condition for a  $\mathbb{G}$ -adapted process to be a  $\mathbb{G}$ -martingale in Section 3. In Section 4, we will first prove Lemma 4.1 and then explicitly describe the  $\mathbb{G}$ -decomposition of a  $(P, \mathbb{F})$ -martingale. In the last section, we obtain the martingale representation theorem for a  $(P^*, \mathbb{G})$ -martingale and then give the martingale representation theorem for a  $(P, \mathbb{G})$ -martingale.

## 2 The setup and notations

We assume that  $\mathbb{F} = \{\mathcal{F}_t; t \geq 0\}$  is a filtration on  $(\Omega, \mathcal{F}, P)$  carrying an  $d$ -dimensional Brownian motion  $W = (W_1, \dots, W_d)'$  as well as an integer-valued random measure  $\mu(du, dy)$  on  $\mathbb{R}_+ \times E$ , where  $(E, \mathcal{E})$  is a Blackwell space.

**Assumption 2.1.** The filtration  $\mathbb{F} = \{\mathcal{F}_t; t \geq 0\}$  is the natural filtration generated by  $W$  and  $\mu$ , i.e.,

$$\mathcal{F}_t = \sigma\{W(s), \mu([0, s] \times A), B; \quad 0 \leq s \leq t, A \in \mathcal{B}, B \in \mathcal{N}\}$$

where  $\mathcal{N}$  is the collection of  $P$ -null sets from  $\mathcal{F}$ .

In the following, we let  $\mathcal{P}(\mathbb{F})$  be the family of all  $\mathbb{F}$ -predictable processes and  $\widetilde{\mathcal{P}}(\mathbb{F}) = \mathcal{P}(\mathbb{F}) \otimes \mathcal{E}$ . We assume that the compensator of  $\mu(du, dy)$  is given by  $\nu(du, dy) = F_u(dy)du$ , where  $F_u(dy)$  is a transition kernel from  $(\Omega \times \mathbb{R}_+, \mathcal{P}(\mathbb{F}))$  to into  $(E, \mathcal{E})$  with  $\int_E F_u(dy) < \infty$ , for each  $u$ . One can see that any local  $(P, \mathbb{F})$ -martingale  $M$  has the form

$$M_t = M_0 + \int_0^t f_1(u)' dW(u) + \int_0^t \int_E f_2(u, y)(\mu(du, dy) - \nu(du, dy)),$$

where  $f_1$  is an  $\mathbb{R}^d$ -valued  $\mathbb{F}$ -predictable process and  $f_2$  is a  $\widetilde{\mathcal{P}}(\mathbb{F})$ -measurable function.

**Remark.** It is easy to see that there exists an  $\mathbb{F}$ -optional process  $\varphi = (\varphi_t)$  and a sequence of stopping times  $(\hat{\tau}_k)$  such that for all positive  $\widetilde{\mathcal{P}}(\mathbb{F})$ -measurable function  $W(\omega, t, y)$ ,

$$\int_0^t \int_E W(\omega, u, y) \mu(\omega; du, dy) = \sum_{(k)} W(\hat{\tau}_k, \varphi_{\hat{\tau}_k}) \mathbb{I}_{\hat{\tau}_k \leq t}.$$

Furthermore, as the compensator of  $\mu(du, dy)$  is  $\nu(du, dy) = F_u(dy)du$ , so the filtration  $\mathbb{F}$  is quasi-left continuous.

Let  $\tilde{\tau}$  be a non-negative random variable and  $L$  be a random variable on  $(\Omega, \mathcal{F})$ , and let the  $\mathbb{G} = \{\mathcal{G}_t; t \geq 0\}$  be the smallest progressive enlargement filtration of  $\mathbb{F}$  such that  $\tilde{\tau}$  is a  $\mathbb{G}$ -stopping time and  $L$  is a  $\mathcal{G}_{\tilde{\tau}}$ -measurable random variable. Let  $\eta(ds, dl)$  be the law of  $(\tilde{\tau}, L)$ , i.e.,  $\eta(ds, dl) = P(\tilde{\tau} \in ds, L \in dl)$ . We need the following assumptions

**Assumption 1.1.** i) the  $\mathbb{F}$ -regular conditional law of  $(\tilde{\tau}, L)$  is equivalent to the law of  $(\tilde{\tau}, L)$ , i.e.,

$$P(\tilde{\tau} \in ds, L \in dl | \mathcal{F}_t) \sim \eta(ds, dl), \quad \text{for every } t \geq 0;$$

ii)  $\eta(ds, dl)$  has no atoms.

**Lemma 2.2.**  $Y_t$  is a  $\mathcal{G}_t$ -measurable random variable (r.v.) if and only if

$$Y_t = y_t^0 1_{t < \tilde{\tau}} + y_t^1(\tilde{\tau}, L) 1_{\tilde{\tau} \leq t},$$

where  $y_t^0$  is a  $\mathcal{F}_t$ -measurable r.v. and  $y_t^1(s, l)$  is an  $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+ \times \mathbb{R})$ -measurable function.

From Pham(2010)<sup>[17]</sup>, we have the following lemma

**Lemma 2.3.** 1) Any  $\mathbb{G}$ -predictable process  $Y = (Y_t)_{t \geq 0}$  is represented as

$$Y_t = Y_t^0 1_{t \leq \tilde{\tau}} + Y^1(\tilde{\tau}, L) 1_{t > \tilde{\tau}},$$

where  $Y^0$  is a  $\mathbb{F}$ -predictable process and where  $Y_t^1(s, l)$  is a  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(R^+) \otimes \mathcal{B}(R)$ -measurable function.

2) Any  $\mathbb{G}$ -optional process  $Y = (Y_t)_{t \geq 0}$  is represented as

$$Y_t = Y_t^0 1_{t < \tilde{\tau}} + Y^1(\tilde{\tau}, L) 1_{t \geq \tilde{\tau}},$$

where  $Y^0$  is a  $\mathbb{F}$ -optional process and where  $Y_t^1(s, l)$  is a  $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(R^+) \otimes \mathcal{B}(R)$ -measurable function.

In this paper, we will consider the following basic questions:

- 1) whether a  $\mathbb{F}$ -martingale is a  $\mathbb{G}$ -martingale and the  $\mathbb{G}$ -decomposition of an  $\mathbb{F}$ -martingale;
- 2) the representation of a  $\mathbb{G}$ -martingale.

From Assumption 1.1, one can see from Jacod(1987)<sup>[10]</sup> or Amendinger(1999)<sup>[1]</sup> that there exists a strictly positive  $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ -measurable function  $(t; \omega; s, l) \rightarrow p_t(\omega; s, l)$ , called the  $(P, \mathbb{F})$ -**conditional density** of  $(\tilde{\tau}, L)$  with respect to  $\eta$ , such that for every  $(s, l) \in \mathbb{R}_+ \times \mathbb{R}$ ,  $p(s, l)$  is a càdlàg  $(P, \mathbb{F})$ -martingale and for any  $B \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ ,

$$P((\tilde{\tau}, L) \in B | \mathcal{F}_t) = \int \int_B p_t(s, l) \eta(ds, dl), \quad \text{for every } t \geq 0, P\text{-a.s.}$$

One can see that  $p_0(s, l) = 1$  for every  $s, l$ .

By the “change of probability measure” viewpoint of Song(1987)<sup>[20]</sup> and similar to Callegaro, Jeanblanc and Zargari(2010)<sup>[5]</sup>, we introduce the filtration  $\mathbb{G}^{\tilde{\tau}, L} = \{\mathcal{G}_t^{\tilde{\tau}, L}; t \geq 0\}$  with  $\mathcal{G}_t^{\tilde{\tau}, L} = \mathcal{F}_t \vee \sigma(\tilde{\tau}, L)$ , one can see that  $\mathbb{G}^{\tilde{\tau}, L}$  is the initial enlargement of the filtration  $\mathbb{F}$  with  $\tilde{\tau}$  and  $L$  and that  $\mathbb{F} \subset \mathbb{G} \subset \mathbb{G}^{\tilde{\tau}, L}$ . Let

$$Z_t = \frac{1}{p_t(\tilde{\tau}, L)},$$

similar to Grorud and Pontier(1998)<sup>[9]</sup> or Amendinger(1999)<sup>[1]</sup>, one can see that  $Z$  is a strictly positive  $(P, \mathbb{G}^{\tilde{\tau}, L})$ -martingale with  $E(Z_t) = 1$ , for every  $t \geq 0$ . Thus one can define a locally equivalent probability measure  $P^*$  by

$$\frac{dP^*}{dP} \Big|_{\mathcal{G}_t^{\tilde{\tau}, L}} = Z_t.$$

Similar to Grorud-Pontier(1998)<sup>[9]</sup> or Amendinger(1999)<sup>[1]</sup>, one can show that

1. under  $P^*$ ,  $(\tilde{\tau}, L)$  is independent of  $\mathcal{F}_t$  for every  $t \geq 0$ ;
2.  $P^*|_{\mathcal{F}_t} = P|_{\mathcal{F}_t}$ ;
3.  $P^*|_{\sigma(\tilde{\tau}, L)} = P|_{\sigma(\tilde{\tau}, L)}$ ,

which implies  $P^*(\tilde{\tau} \in ds, L \in dl | \mathcal{F}_t) = P(\tilde{\tau} \in ds, L \in dl)$ . Similar to Lemma 1.4 in Callegaro, Jeanblanc and Zargari(2010)<sup>[5]</sup>, we have the following lemma

**Lemma 2.4.** 1) Let  $y_t(\tilde{\tau}, L)$  be a  $\mathcal{G}_t^{\tilde{\tau}, L}$ -measurable r.v., then for any  $s \leq t$ ,

$$E_{P^*}(y_t(\tilde{\tau}, L) | \mathcal{G}_s^{\tilde{\tau}, L}) = E_{P^*}(y_t(u, l) | \mathcal{F}_s) \Big|_{u=\tilde{\tau}, l=L};$$

2) if  $y_t(\tilde{\tau}, L)$  is  $P$ -integrable, then

$$E(y_t(\tilde{\tau}, L) | \mathcal{G}_s^{\tilde{\tau}, L}) = \frac{1}{p_s(\tilde{\tau}, L)} E(y_t(u, l) p_t(u, l) | \mathcal{F}_s) \Big|_{u=\tilde{\tau}, l=L}.$$

We have the following Corollaries

**Corollary 2.5** (Characterization of  $(P, \mathbb{G}^{\tilde{\tau}, L})$ -martingales in terms of  $(P, \mathbb{F})$ -martingales). A process  $y_t(\tilde{\tau}, L)$  is a  $(P, \mathbb{G}^{\tilde{\tau}, L})$ -martingale if and only if  $\{y_t(u, l) p_t(u, l); t \geq 0\}$  is a  $(P, \mathbb{F})$ -martingale, for almost every  $u \geq 0, l \in \mathbb{R}$ .

**Corollary 2.6.** *Let  $M = \{M_t; t \geq 0\}$  be a bounded  $(P^*, \mathbb{F})$ -martingale, then  $M$  is a  $(P^*, \mathbb{G}^{\tilde{\tau}, L})$ -martingale and hence a  $(P^*, \mathbb{G})$ -martingale.*

*Proof.* From part 1) of Lemma 2.4, one can see that for any  $s \leq t$

$$\begin{aligned} E_{P^*}(M_t | \mathcal{G}_s^{\tilde{\tau}, L}) &= E_{P^*}(M_t | \mathcal{F}_s) \Big|_{u=\tilde{\tau}, l=L} \\ &= M_s, \end{aligned}$$

thus  $M$  is a  $(P^*, \mathbb{G}^{\tilde{\tau}, L})$ -martingale. Since  $M_s$  is  $\mathcal{F}_s$ -measurable,  $M_s$  is  $\mathcal{G}_s$ -measurable, thus

$$\begin{aligned} E_{P^*}(M_t | \mathcal{G}_s) &= E_{P^*}\{E_{P^*}(M_t | \mathcal{G}_s^{\tilde{\tau}, L}) | \mathcal{G}_s\} \\ &= E_{P^*}\{M_s | \mathcal{G}_s\} \\ &= M_s, \end{aligned}$$

which completes the proof.  $\square$

Let

$$\begin{aligned} G_t &:= P(\tilde{\tau} > t | \mathcal{F}_t) = \int_t^\infty \int_R p_t(s, l) \eta(ds, dl) \quad \text{and} \\ G_t^* &:= P^*(\tilde{\tau} > t | \mathcal{F}_t) = P^*(\tilde{\tau} > t) = P(\tilde{\tau} > t) = \int_0^t \int_R \eta(ds, dl), \end{aligned}$$

from Callegaro, Jeanblanc and Zargari(2010)<sup>[5]</sup>, we find that  $G$  is a  $(P, \mathbb{F})$ -supermartingale and  $G^*$  is a deterministic continuous and decreasing function.

**Theorem 2.7.** *Let  $y_t(\tilde{\tau}, L)$  be a  $\mathcal{G}_t^{\tilde{\tau}, L}$ -measurable  $P$ -integrable r.v., then for  $s \leq t$ ,*

$$E(y_t(\tilde{\tau}, L) | \mathcal{G}_s) = \tilde{y}_s 1_{s < \tilde{\tau}} + \hat{y}_s(\tilde{\tau}, L) 1_{\tilde{\tau} \leq s}$$

with

$$\begin{aligned} \tilde{y}_s &= \frac{1}{G_s} E\left(\int_s^\infty \int_{\mathbb{R}} y_t(u, l) p_t(u, l) \eta(du, dl) \Big| \mathcal{F}_s\right) \\ \hat{y}_s(u, l) &= \frac{1}{p_s(u, l)} E\{y_t(u, l) p_t(u, l) | \mathcal{F}_s\}. \end{aligned}$$

*Proof.* The proof follows from the proof of Lemma 1.5 of Callegaro-Jeanblanc-Zargari(2010)([5]).  $\square$

## 2.1 The $(P, \mathbb{F})$ -density process

Since for any  $s, l$ ,  $p(s, l) = \{p_t(s, l); t \geq 0\}$  is a strictly positive  $(P, \mathbb{F})$ -martingale,  $\{p(s, l); s \geq 0, l \in \mathbb{R}\}$  is a  $(P, \mathbb{F})$ -martingale system. Since  $\{p_t(s, l); t \geq 0\}$  is a strictly positive martingale, one can see that  $p_t(s, l)$  can be represented in the following form

$$\begin{aligned} p_t(s, l) &= E(p_s(s, l) | \mathcal{F}_t) \mathbb{I}_{t < s} + p_s(s, l) \exp \left\{ \int_s^t \theta_1(u; s, l)' dW(u) - \frac{1}{2} \int_s^t \|\theta_1(u; s, l)\|^2 du \right\} \\ &\quad \times \exp \left\{ \int_s^t \int_E \{ \ln(1 + \theta_2(u, y; s, l)) \} \mu(du, dy) - \int_s^t \int_E \theta_2(u, y; s, l) \nu(du, dy) \right\} \mathbb{I}_{t \geq s}, \end{aligned} \tag{2.1}$$

which implies that  $p_t(s, l)$  is completely determined by  $p_s(s, l)$ ,  $\theta_1(u; s, l)\mathbb{I}_{u>s}$  and  $\theta_2(u, y; s, l)\mathbb{I}_{u>s}$ . Since  $E(p_s(s, l)|\mathcal{F}_t)$  can be written in the following form

$$\begin{aligned} E(p_s(s, l)|\mathcal{F}_t) &= 1 + \int_0^{t \wedge s} p_{u-}(s, l)\theta_1^*(u; s, l)'dW(u) \\ &\quad + \int_0^{t \wedge s} \int_E p_{u-}(s, l)\theta_2^*(u, y; s, l)\{\mu(du, dy) - \nu(du, dy)\}, \end{aligned}$$

hence  $p_t(s, l)$  can also be represented in the following way

$$\begin{aligned} p_t(s, l) &= 1 + \int_0^t p_{u-}(s, l)\{\theta_1^*(u; s, l)\mathbb{I}_{u \leq s} + \theta_1(u; s, l)\mathbb{I}_{u > s}\}'dW(u) \\ &\quad + \int_0^t \int_E p_{u-}(s, l)\{\theta_2^*(u, y; s, l)\mathbb{I}_{u \leq s} + \theta_2(u, y; s, l)\mathbb{I}_{u > s}\}\{\mu(du, dy) - \nu(du, dy)\}. \end{aligned} \quad (2.2)$$

First, we noted that the following equation by Fubini theorem.

$$\int_t^\infty \int_{\mathbb{R}} E(p_s(s, l)|\mathcal{F}_t)\eta(ds, dl) = E\left(\int_t^\infty \int_{\mathbb{R}} p_s(s, l)\eta(ds, dl) \middle| \mathcal{F}_t\right). \quad (2.3)$$

We first give an examples before characterizing the conditional density process.

**Example 2.8.** Let  $\theta_1(u; s, l)\mathbb{I}_{u>s} := 0$  and  $\theta_2(u, y; s, l)\mathbb{I}_{u>s} := 0$ , then

$$p_t(s, l) = E(p_s(s, l)|\mathcal{F}_t)\mathbb{I}_{t < s} + p_s(s, l)\mathbb{I}_{t \geq s}.$$

Since  $\int_0^\infty \int_{\mathbb{R}} p_t(s, l)\eta(ds, dl) \equiv 1$  for each  $t$ , one can see from (2.3) that

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}} p_t(s, l)\eta(ds, dl) \\ &= E\left(\int_t^\infty \int_{\mathbb{R}} p_s(s, l)\eta(ds, dl) \middle| \mathcal{F}_t\right) + \int_0^t \int_{\mathbb{R}} p_s(s, l)\eta(ds, dl) \\ &= E\left(\int_0^\infty \int_{\mathbb{R}} p_s(s, l)\eta(ds, dl) \middle| \mathcal{F}_t\right) = 1, \end{aligned}$$

for every  $t \geq 0$ . Let  $t \rightarrow \infty$ , one can see that  $p_s(s, l)$  must satisfy the following condition

$$\int_0^\infty \int_{\mathbb{R}} p_s(s, l)\eta(ds, dl) = 1.$$

More generally, we have the following theorem

**Theorem 2.9.** For any given bounded  $p_s(s, l)$ ,  $\theta_1(u; s, l)\mathbb{I}_{u>s}$  and  $\theta_2(u, y; s, l)\mathbb{I}_{u>s}$ , let

$$\begin{aligned} Z_{s,l}^{\theta_1, \theta_2}(t) &= \exp\left\{\int_s^t \theta_1(u; s, l)'dW(u) - \frac{1}{2} \int_s^t \|\theta_1(u; s, l)\|^2 du\right\} \\ &\quad \times \exp\left\{\int_s^t \int_E \{\ln(1 + \theta_2(u, y; s, l))\}\mu(du, dy) - \int_s^t \int_E \theta_2(u, y; s, l)\nu(du, dy)\right\}. \end{aligned}$$

If

$$p_t(s, l) = E(p_s(s, l)|\mathcal{F}_t)\mathbb{I}_{t < s} + p_s(s, l)Z_{s,l}^{\theta_1, \theta_2}(t)\mathbb{I}_{t \geq s}$$

is the density process of a pair  $(\tilde{\tau}, L)$  with respect to  $(P, \mathbb{F})$ , then  $p_s(s, l)$ ,  $\theta_1(u; s, l)\mathbb{I}_{u>s}$  and  $\theta_2(u, y; s, l)\mathbb{I}_{u>s}$  satisfies the following condition

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}} p_s(s, l) \eta(ds, dl) &= 1 - \int_0^\infty \left\{ \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_1(u; s, l) \eta(ds, dl) \right\}' dW(u) \\ &\quad - \int_0^\infty \int_E \left\{ \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_2(u, y; s, l) \eta(ds, dl) \right\} \{ \mu(du, dy) - \nu(du, dy) \}. \end{aligned} \quad (2.4)$$

*Proof.* One can see that

$$\begin{aligned} Z_{s,l}^{\theta_1, \theta_2}(t) &= 1 + \int_s^t Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_1(u; s, l)' dW(u) \\ &\quad + \int_s^t \int_E Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_2(u, y; s, l) \{ \mu(du, dy) - \nu(du, dy) \}, \end{aligned}$$

From  $\int_0^\infty \int_{\mathbb{R}} p_t(s, l) \eta(ds, dl) \equiv 1$  for each  $t$ , one derives from (2.3) that

$$\begin{aligned} 1 &= \int_0^\infty \int_{\mathbb{R}} p_t(s, l) \eta(ds, dl) \\ &= E \left( \int_0^\infty \int_{\mathbb{R}} p_s(s, l) \eta(ds, dl) \middle| \mathcal{F}_t \right) \\ &\quad - \int_0^t \int_{\mathbb{R}} p_s(s, l) \eta(ds, dl) + \int_0^t \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(t) \eta(ds, dl) \\ &= E \left( \int_0^\infty \int_{\mathbb{R}} p_s(s, l) \eta(ds, dl) \middle| \mathcal{F}_t \right) \\ &\quad + \int_0^t \int_{\mathbb{R}} p_s(s, l) \left\{ \int_s^t Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_1(u; s, l)' dW(u) \right. \\ &\quad \left. + \int_s^t \int_E Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_2(u, y; s, l) \{ \mu(du, dy) - \nu(du, dy) \} \right\} \eta(ds, dl) \\ &= E \left( \int_0^\infty \int_{\mathbb{R}} p_s(s, l) \eta(ds, dl) \middle| \mathcal{F}_t \right) \\ &\quad + \int_0^t \left\{ \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_1(u; s, l) \eta(ds, dl) \right\}' dW(u) \\ &\quad + \int_0^t \int_E \left\{ \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_2(u, y; s, l) \eta(ds, dl) \right\} \{ \mu(du, dy) - \nu(du, dy) \}, \end{aligned}$$

where the last equality comes from the stochastic Fubini theorem for general semimartingales, see Theorem 4.1.1 of Jeanblanc, Yor and Chesney(2009) <sup>[11]</sup>.

□

**Corollary 2.10.** *Under the conditions of Theorem 2.9, the survival process of  $\tilde{\tau}$  with respect to  $(P, \mathbb{F})$  is given by*

$$\begin{aligned} G_t &= 1 - \int_0^t \int_{\mathbb{R}} p_s(s, l) \eta(ds, dl) \\ &\quad - \int_0^t \left\{ \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_1(u; s, l) \eta(ds, dl) \right\}' dW(u) \\ &\quad - \int_0^t \int_E \left\{ \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_2(u, y; s, l) \eta(ds, dl) \right\} \{ \mu(du, dy) - \nu(du, dy) \}. \end{aligned} \quad (1.2)$$



*Proof.* From the definition of  $G_t$ , one sees that

$$\begin{aligned}
G_t &= P(\tilde{\tau} > t | \mathcal{F}_t) = \int_t^\infty \int_{\mathbb{R}} p_t(s, l) \eta(ds, dl) \\
&= \int_t^\infty \int_{\mathbb{R}} E[p_s(s, l) | \mathcal{F}_t] \eta(ds, dl) \\
&= E \left[ \int_t^\infty \int_{\mathbb{R}} p_s(s, l) \eta(ds, dl) \middle| \mathcal{F}_t \right] \\
&= - \int_0^t \int_{\mathbb{R}} p_s(s, l) \eta(ds, dl) + E \left[ \int_0^\infty \int_{\mathbb{R}} p_s(s, l) \eta(ds, dl) \middle| \mathcal{F}_t \right] \\
&= 1 - \int_0^t \int_{\mathbb{R}} p_s(s, l) \eta(ds, dl) \\
&\quad - \int_0^t \left\{ \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_1(u; s, l) \eta(ds, dl) \right\}' dW(u) \\
&\quad - \int_0^t \int_E \left\{ \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_2(u, y; s, l) \eta(ds, dl) \right\} \{ \mu(du, dy) - \nu(du, dy) \},
\end{aligned}$$

which completes the proof.  $\square$

**Remark.** We note here that (1.2) does not directly depend on  $\theta_1^*(u; s, l)$  and  $\theta_2^*(u; s, l)$ , that is to say, the survival process  $G$  is completely determined by  $p_s(s, l)$ ,  $\theta_1(u; s, l) \mathbb{I}_{u > s}$  and  $\theta_2(u; s, l) \mathbb{I}_{u > s}$ .

**Remark.** From Corollary 2.14, one can this if and only if  $\theta_1 = \theta_2 = 0$  and the survival process  $G$  of  $\tilde{\tau}$  with respect to  $(P, \mathbb{F})$  is a decreasing function, i.e, the  $\mathbb{F}$  is immersed in  $\mathbb{G}$ .

### 3 $\mathbb{G}$ -martingales' characterization

In this section, we will characterize  $(P, \mathbb{G})$ -martingales in terms of  $(P, \mathbb{F})$ -martingales. Similar to Proposition 2.2 of [5], we have the theorem

**Theorem 3.1** (Characterization of  $(P, \mathbb{G})$ -martingales in terms of  $(P, \mathbb{F})$ -martingales). *Let  $y = \{y_t; t \geq 0\}$ , where  $y_t := \tilde{y}_t 1_{t < \tilde{\tau}} + \hat{y}_t(\tilde{\tau}, L) 1_{t \geq \tilde{\tau}}$ , be a  $\mathbb{G}$ -adapted process, then  $y$  is a  $(P, \mathbb{G})$ -martingale if and only if the following two conditions are satisfied:*

- (i) *for  $\eta$ -almost every  $u \geq 0$  and  $l \in \mathbb{R}$ ,  $\{\hat{y}_t(u; l) p_t(u; l); t \geq u\}$  is a  $(P, \mathbb{F})$ -martingale;*
- (ii) *the process  $\{\tilde{y}_t G_t + \int_0^t \int_{\mathbb{R}} \hat{y}_u(u, l) p_u(u, l) \eta(du, dl); t \geq 0\}$  is a  $(P, \mathbb{F})$ -martingale.*

To give a proof, we need the following lemma

**Lemma 3.2** (Projection). *Let  $\mathbb{F}^*$  be a filtration larger than  $\mathbb{F}$ , i.e.,  $\mathbb{F} \subset \mathbb{F}^*$ . If  $x$  is a uniformly integrable (u.i.)  $\mathbb{F}$ -martingale, then there exists an  $\mathbb{F}^*$ -martingale  $x^*$  such that  $E(x_t^* | \mathcal{F}_t) = x_t$ .*

*Proof of Theorem 3.1.* Let  $y$  is a u.i.  $(P, \mathbb{G})$ -martingale (otherwise consider the case of stopped martingale as shown in [5]), since  $\mathbb{F} \subset \mathbb{G} \subset \mathbb{G}^{\tilde{\tau}, L}$ , one can see that there exists a  $\mathbb{G}^{\tilde{\tau}, L}$ -martingale  $y^*(\tilde{\tau}, L)$  such that  $y_t = E(y_t^*(\tilde{\tau}, L) | \mathcal{G}_t)$ , i.e., for  $\eta$ -almost every  $u \geq 0$  and  $l \in \mathbb{R}$ , the process  $\{y_t^*(u, l) p_t(u, l); t \geq 0\}$  is a u.i.  $(P, \mathbb{F})$ -martingale. For any  $t \geq 0$ , since  $y_t^*(\tilde{\tau}, L) 1_{t \geq \tilde{\tau}}$  is  $\mathcal{G}_t$ -measurable and  $1_{t \geq \tilde{\tau}}$  is  $\mathcal{G}_t$ -measurable, one can see that

$$y_t^*(\tilde{\tau}, L) 1_{t \geq \tilde{\tau}} = E[y_t^*(\tilde{\tau}, L) | \mathcal{G}_t] 1_{t \geq \tilde{\tau}} = y_t 1_{t \geq \tilde{\tau}} = \hat{y}_t(\tilde{\tau}, L) 1_{t \geq \tilde{\tau}},$$

which implies that  $y_t^*(u, l)1_{t \geq u} = \widehat{y}(u, l)1_{t \geq u}$ ,  $\eta$ -a.s., for every  $t \geq 0$ , and hence  $\{\widehat{y}_t(u, l)p_t(u, l); t \geq u\}$  is a  $(P, \mathbb{F})$ -martingale and (i) is proved. Furthermore, from Theorem 2.7, one can see that

$$E(y_t^*(\tilde{\tau}, L)|\mathcal{G}_t) = \mathbb{I}_{t < \tilde{\tau}} \frac{1}{G_t} E\left(\int_t^\infty \int_{\mathbb{R}} y_t^*(u, l)p_t(u, l)\eta(du, dl) \middle| \mathcal{F}_t\right) + \widehat{y}_t(\tilde{\tau}, L)\mathbb{I}_{\tilde{\tau} \leq t},$$

thus

$$\begin{aligned} \tilde{y}_t \mathbb{I}_{t < \tilde{\tau}} &= y_t \mathbb{I}_{t < \tilde{\tau}} = E(y_t^*(\tilde{\tau}, L)|\mathcal{G}_t)\mathbb{I}_{t < \tilde{\tau}} = \mathbb{I}_{t < \tilde{\tau}} \frac{1}{G_t} E\left(\int_t^\infty \int_{\mathbb{R}} y_t^*(u, l)p_t(u, l)\eta(du, dl) \middle| \mathcal{F}_t\right) \\ &= \mathbb{I}_{t < \tilde{\tau}} \frac{1}{G_t} E\left(\int_t^\infty \int_{\mathbb{R}} y_u^*(u, l)p_u(u, l)\eta(du, dl) \middle| \mathcal{F}_t\right), \end{aligned}$$

where the last equality results from the  $(P, \mathbb{F})$ -martingale property of the process  $y^*(u, l)p(u, l)$ , for  $\eta$ -almost every  $u \geq 0$ ,  $l \geq 0$ . We deduce

$$\begin{aligned} \tilde{y}_t G_t &= E\left(\int_t^\infty \int_{\mathbb{R}} y_u^*(u, l)p_u(u, l)\eta(du, dl) \middle| \mathcal{F}_t\right) \\ &= E\left(\int_0^\infty \int_{\mathbb{R}} y_u^*(u, l)p_u(u, l)\eta(du, dl) \middle| \mathcal{F}_t\right) - \int_0^t \int_{\mathbb{R}} y_u^*(u, l)p_u(u, l)\eta(du, dl), \end{aligned}$$

which implies that  $\left\{\tilde{y}_t G_t + \int_0^t \int_{\mathbb{R}} \widehat{y}_u(u, l)p_u(u, l)\eta(du, dl), 0 \leq t \leq \infty\right\}$  is a  $(P, \mathbb{F})$ -martingale (since  $y_u^*(u, l) = \widehat{y}_u(u, l)$ ) and (ii) immediately follows.

Conversely, assuming (i) and (ii), we verify  $E(y_t|\mathcal{G}_s) = y_s$  for  $s \leq t$ . Indeed,

$$\begin{aligned} E(y_t|\mathcal{G}_s) &= E(\mathbb{I}_{t < \tilde{\tau}} \tilde{y}_t + \mathbb{I}_{s < \tilde{\tau} \leq t} \widehat{y}_t(\tilde{\tau}, L)|\mathcal{G}_s) + E(\mathbb{I}_{\tilde{\tau} \leq s} \widehat{y}_t(\tilde{\tau}, L)|\mathcal{G}_s) \\ &= \mathbb{I}_{s < \tilde{\tau}} \frac{1}{G_s} E(\mathbb{I}_{t < \tilde{\tau}} \tilde{y}_t + \mathbb{I}_{s < \tilde{\tau} \leq t} \widehat{y}_t(\tilde{\tau}, L)|\mathcal{F}_s) + \frac{1}{p_s(\tilde{\tau}, L)} \mathbb{I}_{\tilde{\tau} \leq s} E(\widehat{y}_t(u, l)p_t(u, l)|\mathcal{F}_s)|_{u=\tilde{\tau}, l=L}. \end{aligned}$$

where we also use Lemma 3.1.2 and Lemma 1.5 in Bielecki, Jeanblanc and Rutkowski [4] to obtain the last equality. Next, using condition (i), it follows that

$$\begin{aligned} E(y_t|\mathcal{G}_s) &= \mathbb{I}_{s < \tilde{\tau}} \frac{1}{G_s} E(\tilde{y}_t G_t + \int_s^t \int_{\mathbb{R}} \widehat{y}_u(u, l)p_t(u, l)\eta(du, dy)|\mathcal{F}_s) + \mathbb{I}_{\tilde{\tau} \leq s} \frac{1}{p_s(\tilde{\tau}, L)} \widehat{y}_s(\tilde{\tau}, L)p_s(\tilde{\tau}, L) \\ &= \mathbb{I}_{s < \tilde{\tau}} \frac{1}{G_s} E(\tilde{y}_t G_t + \int_0^t \int_{\mathbb{R}} \widehat{y}_u(u, l)p_u(u, l)\eta(du, dy)|\mathcal{F}_s) \\ &\quad - \mathbb{I}_{s < \tilde{\tau}} \frac{1}{G_s} \int_0^s \int_{\mathbb{R}} \widehat{y}_u(u, l)p_u(u, l)\eta(du, dl) + \mathbb{I}_{\tilde{\tau} \leq s} \widehat{y}_s(\tilde{\tau}, L) \\ &= \mathbb{I}_{s < \tilde{\tau}} \frac{1}{G_s} \tilde{y}_s G_s + \mathbb{I}_{\tilde{\tau} \leq s} \widehat{y}_s(\tilde{\tau}, L) \\ &= y_s, \end{aligned}$$

where we used condition (ii) to obtain the next-to-last identity.  $\square$

Similarly, we have the following corollary

**Corollary 3.3.** *Let  $y = \{y_t; t \geq 0\}$ , where  $y_t := \tilde{y}_t 1_{t < \tilde{\tau}} + \widehat{y}_t(\tilde{\tau}, L) 1_{t \geq \tilde{\tau}}$ , be a  $\mathbb{G}$ -adapted process, then  $y$  is a  $(P^*, \mathbb{G})$ -martingale if and only if the following two conditions are satisfied:*

- (i) *for  $\eta$ -almost every  $u \geq 0$  and  $l \in \mathbb{R}$ ,  $\{\widehat{y}_t(u, l); t \geq u\}$  is a  $(P^*, \mathbb{F})$ -martingale;*
- (ii) *the process  $\{\tilde{y}_t G_t^* + \int_0^t \int_{\mathbb{R}} \widehat{y}_u(u, l)\eta(du, dl); t \geq 0\}$  is a  $(P^*, \mathbb{F})$ -martingale.*

## 4 Canonical decomposition of a $(P, \mathbb{F})$ -martingale in $(P, \mathbb{G})$

We now consider the canonical decomposition of any  $(P, \mathbb{F})$  martingale  $m$  in the enlarged filtration  $\mathbb{G}$  respectively under Assumption 1.1. From Theorem 2.9, one can see that  $p_t(s, l)$  can be determined by  $p_s(s, l)$ ,  $\theta_1(u; s, l)\mathbb{I}_{u>s}$  and  $\theta_2(u, y; s, l)\mathbb{I}_{u>s}$ . We have

$$p_t(s, l) = E[p_s(s, l) | \mathcal{F}_t] \mathbb{I}_{t < s} + p_s(s, l) Z_{s, l}^{\theta_1, \theta_2}(t) \mathbb{I}_{t \geq s}$$

and

$$\begin{aligned} p_t(s, l) \mathbb{I}_{t \geq s} &= p_s(s, l) \mathbb{I}_{t \geq s} + \int_0^t p_{u-}(s, l) \theta_1(u; s, l)' \mathbb{I}_{u>s} dW(u) \\ &\quad + \int_0^t \int_E p_{u-}(s, l) \theta_2(u, y; s, l) \mathbb{I}_{u>s} \{\mu(du, dy) - \nu(du, dy)\}, \end{aligned}$$

thus

$$\begin{aligned} p_t(\tilde{\tau}, L) \mathbb{I}_{t \geq \tilde{\tau}} &= p_{\tilde{\tau}}(\tilde{\tau}, L) \mathbb{I}_{t \geq \tilde{\tau}} + \int_0^t p_{u-}(\tilde{\tau}, L) \theta_1(u; \tilde{\tau}, L)' \mathbb{I}_{u>\tilde{\tau}} dW(u) \\ &\quad + \int_0^t \int_E p_{u-}(\tilde{\tau}, L) \theta_2(u, y; \tilde{\tau}, L) \mathbb{I}_{u>\tilde{\tau}} \{\mu(du, dy) - \nu(du, dy)\}. \end{aligned}$$

Recall that  $G_t^* = P(\tilde{\tau} > t)$  is a deterministic continuous function satisfying  $0 < G_t^* < 1$  for each  $t \in (0, \infty)$ , there are no atoms. To obtain the canonical decomposition of a  $(P, \mathbb{F})$  martingale in the filtration  $\mathbb{G}$ , we need the following lemma:

**Lemma 4.1.** *For any positive  $\mathcal{O}(\mathbb{F}) \times \mathcal{B}$ -measurable function  $f(s, l)$  such that  $E_{P^*}(|f(\tilde{\tau}, L)|) < \infty$ , let*

$$A_t^{f,*} := \int_0^t \int_{\mathbb{R}} \frac{f(s, l)}{G_{s-}^*} \eta(ds, dl),$$

*then  $A^{f,*}$  is a continuous increasing  $\mathbb{F}$ -adapted process and*

$$M_t^{f,*} = f(\tilde{\tau}, L) \mathbb{I}_{t \geq \tilde{\tau}} - A_{t \wedge \tilde{\tau}}^{f,*}$$

*is a  $(P^*, \mathbb{G})$ -martingale, i.e.,  $A_{\cdot \wedge \tilde{\tau}}^{f,*}$  is the  $(P^*, \mathbb{G})$ -compensator of  $f(\tilde{\tau}, L) \mathbb{I}_{t \geq \tilde{\tau}}$ .*

*Proof.* For any  $t_1 < t_2$ , we have

$$E_{P^*}[f(\tilde{\tau}, L) \mathbb{I}_{\tilde{\tau} \leq t} | \mathcal{F}_t] = \int_0^t \int_{\mathbb{R}} f(s, l) \eta(ds, dl). \quad (4.1)$$

In fact, one can see from the independence of  $(\tilde{\tau}, L)$  with respect to  $\mathcal{F}_t$  under  $P^*$  that (4.1) holds for all positive  $\mathcal{B}(\mathbb{R}^+ \times \mathbb{R})$ -measurable functions  $f(s, l)$ . In general, one gets from the independence lemma (see Lemma 2.3.4 of Shreve(2003)<sup>[19]</sup>) and the monotone class theorem that (4.1) still holds for any positive  $\mathcal{O}(\mathbb{F}) \times \mathcal{B}$ -measurable function  $f(s, l)$ . And since there are no atoms, one can see that  $\int_0^t \int_{\mathbb{R}} f(s, l) \eta(ds, dl)$  is a continuous increasing  $\mathbb{F}$ -adapted process, thus a  $\mathbb{F}$ -predictable process.

For any  $t_1 < t_2$ , we have

$$E_{P^*}[f(\tilde{\tau}, L) \mathbb{I}_{t_1 < \tilde{\tau} \leq t_2} | \mathcal{F}_{t_1}] = E_{P^*}\left[\int_{t_1}^{t_2} \int_{\mathbb{R}} f(s, l) \eta(ds, dl) \middle| \mathcal{F}_{t_1}\right]$$

and that

$$\begin{aligned}
& E_{P^*} \left[ \int_{t_1}^{t_2} \int_{\mathbb{R}} \frac{\mathbb{I}_{s \leq \tilde{\tau}}}{G_s^*} f(s, l) \eta(ds, dl) \middle| \mathcal{F}_{t_1} \right] \\
&= E_{P^*} \left[ \int_{t_1}^{t_2} \int_{\mathbb{R}} \frac{E_{P^*}[\mathbb{I}_{s \leq \tilde{\tau}} | \mathcal{F}_s]}{G_{s-}^*} f(s, l) \eta(ds, dl) \middle| \mathcal{F}_{t_1} \right] \\
&= E_{P^*} \left[ \int_{t_1}^{t_2} \int_{\mathbb{R}} f(s, l) \eta(ds, dl) \middle| \mathcal{F}_{t_1} \right],
\end{aligned}$$

hence

$$\begin{aligned}
& E_{P^*} [M_{t_2}^{f,*} - M_{t_1}^{f,*} | \mathcal{G}_{t_1}] = E_{P^*} \left[ f(\tilde{\tau}, L) \mathbb{I}_{t_1 < \tilde{\tau} \leq t_2} - \int_{t_1}^{t_2} \int_{\mathbb{R}} \frac{\mathbb{I}_{s \leq \tilde{\tau}}}{G_s^*} f(s, l) \eta(ds, dl) \middle| \mathcal{G}_{t_1} \right] \\
&= \frac{1}{G_{t_1}^*} \left\{ E_{P^*} [f(\tilde{\tau}, L) \mathbb{I}_{t_1 < \tilde{\tau} \leq t_2} | \mathcal{F}_{t_1}] \right. \\
&\quad \left. - E_{P^*} \left[ \int_{t_1}^{t_2} \int_{\mathbb{R}} \frac{\mathbb{I}_{s \leq \tilde{\tau}}}{G_s^*} f(s, l) \eta(ds, dl) \middle| \mathcal{F}_{t_1} \right] \right\} \mathbb{I}_{t_1 < \tilde{\tau}} \\
&= 0,
\end{aligned}$$

thus  $E_{P^*}[M_{t_2}^{f,*} | \mathcal{G}_{t_1}] = M_{t_1}^{f,*}$  and  $M^{f,*}$  is a  $(P^*, \mathbb{G})$ -martingale.  $\square$

We need the following assumption thereafter.

**Assumption 4.2.** We assume that  $E_{P^*}(p_{\tilde{\tau}}(\tilde{\tau}, L)) < \infty$  and  $E_{P^*}\left(\frac{G_{\tilde{\tau}}}{G_{\tilde{\tau}}^*}\right) < \infty$ .

**Corollary 4.3.** Suppose Assumption 4.2 holds. Let

$$N_1(t) := p_{\tilde{\tau}}(\tilde{\tau}, L) \mathbb{I}_{t \geq \tilde{\tau}} - \int_0^{t \wedge \tilde{\tau}} \int_{\mathbb{R}} \frac{1}{G_s^*} p_s(s, l) \eta(ds, dl),$$

then  $\{N_1(t); t \geq 0\}$  is a uniformly integrable  $(P^*, \mathbb{G})$ -martingale.

**Corollary 4.4.** Let Assumption 4.2 holds and let

$$N_2(t) := \frac{G_{\tilde{\tau}}}{G_{\tilde{\tau}}^*} \mathbb{I}_{t \geq \tilde{\tau}} + \int_0^{t \wedge \tilde{\tau}} \frac{G_u}{(G_u^*)^2} dG_u^*,$$

then  $\{N_2(t); t \geq 0\}$  is a uniformly integrable  $(P^*, \mathbb{G})$ -martingale.

*Proof.* One can see that

$$\begin{aligned}
N_2(t) &= \frac{G_{\tilde{\tau}}}{G_{\tilde{\tau}}^*} \mathbb{I}_{t \geq \tilde{\tau}} + \int_0^{t \wedge \tilde{\tau}} \frac{G_u}{(G_u^*)^2} dG_u^* \\
&= \frac{G_{\tilde{\tau}}}{G_{\tilde{\tau}}^*} \mathbb{I}_{t \geq \tilde{\tau}} - \int_0^{t \wedge \tilde{\tau}} \int_{\mathbb{R}} \frac{1}{G_u^*} \frac{G_u}{G_u^*} \eta(du, dl),
\end{aligned}$$

since  $G_u^* = P^*(\tilde{\tau} > u) = \int_u^\infty \int_{\mathbb{R}} \eta(du, dl)$ . From Lemma 4.1, one can see that  $N_2$  is a uniformly integrable  $(P^*, \mathbb{G})$ -martingale.  $\square$

**Theorem 4.5.** Assume Assumption 4.2 holds. Let  $Z_t^* := \frac{G_t}{G_t^*} \mathbb{I}_{t < \tilde{\tau}} + p_t(\tilde{\tau}, L) \mathbb{I}_{t \geq \tilde{\tau}}$ , then  $Z^*$  is a uniformly integrable  $(P^*, \mathbb{G})$ -martingale with the following decomposition

$$\begin{aligned} Z_t^* &= 1 - \int_0^t Z_{u-}^* \left\{ \frac{1}{G_{u-}} \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_1(u; s, l) \eta(ds, dl) \mathbb{I}_{u \leq \tilde{\tau}} \right. \\ &\quad \left. - \theta_1(u; \tilde{\tau}, L) \mathbb{I}_{u > \tilde{\tau}} \right\}' dW(u) \\ &\quad - \int_0^t \int_E Z_{u-}^* \left\{ \frac{1}{G_{u-}} \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_2(u, y; s, l) \eta(ds, dl) \mathbb{I}_{u \leq \tilde{\tau}} \right. \\ &\quad \left. - \theta_2(u, y; \tilde{\tau}, L) \mathbb{I}_{u > \tilde{\tau}} \right\} \{ \mu(du, dy) - \nu(du, dy) \} \\ &\quad + N_1(t) - N_2(t). \end{aligned}$$

*Proof.* One can see that  $Z_t^* = \frac{1}{E\left[\frac{1}{p_t(\tilde{\tau}, L)} \middle| \mathcal{G}_t\right]}$ , from which  $Z^*$  is a uniformly integrable

$(P^*, \mathbb{G})$ -martingale. Since

$$\begin{aligned} \frac{G_t}{G_t^*} &= 1 - \int_0^t \int_{\mathbb{R}} \frac{1}{G_s^*} p_s(s, l) \eta(ds, dl) \\ &\quad - \int_0^t \frac{1}{G_u^*} \left\{ \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_1(u; s, l) \eta(ds, dl) \right\}' dW(u) \\ &\quad - \int_0^t \int_E \frac{1}{G_u^*} \left\{ \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_2(u, y; s, l) \eta(ds, dl) \right\} \{ \mu(du, dy) - \nu(du, dy) \} \\ &\quad - \int_0^t \frac{G_{u-}}{(G_u^*)^2} dG_u^*, \end{aligned}$$

one can see that

$$\begin{aligned} Z_t^* &= 1 - \int_0^{t \wedge \tilde{\tau}} \int_{\mathbb{R}} \frac{1}{G_s^*} p_s(s, l) \eta(ds, dl) \\ &\quad - \int_0^{t \wedge \tilde{\tau}} \frac{1}{G_u^*} \left\{ \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_1(u; s, l) \eta(ds, dl) \right\}' dW(u) \\ &\quad - \int_0^{t \wedge \tilde{\tau}} \int_E \frac{1}{G_u^*} \left\{ \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_2(u, y; s, l) \eta(ds, dl) \right\} \{ \mu(du, dy) - \nu(du, dy) \} \\ &\quad - \int_0^{t \wedge \tilde{\tau}} \frac{G_{u-}}{(G_u^*)^2} dG_u^* - \frac{G_{\tilde{\tau}-}}{G_{\tilde{\tau}}^*} \mathbb{I}_{t \geq \tilde{\tau}} \\ &\quad + p_{\tilde{\tau}}(\tilde{\tau}, L) \mathbb{I}_{t \geq \tilde{\tau}} + \int_0^t p_{u-}(\tilde{\tau}, L) \theta_1(u; \tilde{\tau}, L)' \mathbb{I}_{u > \tilde{\tau}} dW(u) \\ &\quad + \int_0^t \int_E p_{u-}(\tilde{\tau}, L) \theta_2(u, y; \tilde{\tau}, L) \mathbb{I}_{u > \tilde{\tau}} \{ \mu(du, dy) - \nu(du, dy) \} \end{aligned}$$

$$\begin{aligned}
&= 1 - \int_0^t Z_{u-}^* \left\{ \frac{1}{G_{u-}} \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_1(u; s, l) \eta(ds, dl) \mathbb{I}_{u \leq \tilde{\tau}} \right. \\
&\quad \left. - \theta_1(u; \tilde{\tau}, L) \mathbb{I}_{u > \tilde{\tau}} \right\}' dW(u) \\
&\quad - \int_0^t \int_E Z_{u-}^* \left\{ \frac{1}{G_{u-}} \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_2(u, y; s, l) \eta(ds, dl) \mathbb{I}_{u \leq \tilde{\tau}} \right. \\
&\quad \left. - \theta_2(u, y; \tilde{\tau}, L) \mathbb{I}_{u > \tilde{\tau}} \right\} \{ \mu(du, dy) - \nu(du, dy) \} \\
&\quad - \int_0^{t \wedge \tilde{\tau}} \frac{G_{u-}}{(G_u^*)^2} dG_u^* - \frac{G_{\tilde{\tau}-}}{G_{\tilde{\tau}}^*} \mathbb{I}_{t \geq \tilde{\tau}} \\
&\quad + p_{\tilde{\tau}}(\tilde{\tau}, L) \mathbb{I}_{t \geq \tilde{\tau}} - \int_0^{t \wedge \tilde{\tau}} \int_{\mathbb{R}} \frac{1}{G_s^*} p_s(s, l) \eta(ds, dl),
\end{aligned}$$

which completes the proof.  $\square$

By Theorem 4.5, we give the  $\mathbb{G}$ -decomposition of a  $(P, \mathbb{F})$  martingale as follows:

**Theorem 4.6.** *Let  $p_s(s, l)$ ,  $\theta_1(u; s, l) \mathbb{I}_{u > s}$  and  $\theta_2(u, y; s, l) \mathbb{I}_{u > s}$  be given as in Theorem 2.9. If  $m$  is a càdlàg  $(P, \mathbb{F})$ -local martingale of the following form*

$$m_t = m_0 + \int_0^t \xi_1(u)' dW(u) + \int_0^t \int_E \xi_2(u, y) \{ \mu(du, dy) - \nu(du, dy) \},$$

then under Assumption 4.2,

$$\begin{aligned}
X_t := m_t + \int_0^{t \wedge \tilde{\tau}} \frac{1}{G_{u-}} &\left\{ \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \left\{ \xi_1(u)' \theta_1(u; s, l) \right. \right. \\
&\quad \left. \left. + \int_E \theta_2(u, y; s, l) \xi_2(u, y) F_u(dy) \right\} \eta(ds, dl) \right\} du \\
&- \int_{\tilde{\tau}}^t \left\{ \xi_1(u)' \theta_1(u; \tilde{\tau}, L) + \int_E \theta_2(u, y; \tilde{\tau}, L) \xi_2(u, y) F_u(dy) \right\} du,
\end{aligned}$$

is a  $(P, \mathbb{G})$ -local martingale.

**Remark.**

- 1) Theorem 4.6 may be viewed as a corollary of Callegaro, Jeanblanc and Zargari(2010)<sup>[5]</sup> and El Karoui, Jeanblanc and Jiao(2009)<sup>[8]</sup>, the main difference is that the decomposition of  $\mathbb{F}$ -local martingale in  $\mathbb{G}$  we give here only depends on  $p_s(s, l)$ ,  $\theta_1(u; s, l) \mathbb{I}_{u > s}$  and  $\theta_2(u, y; s, l) \mathbb{I}_{u > s}$ , since in the integral

$$\int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \left\{ \xi_1(u)' \theta_1(u; s, l) + \int_E \theta_2(u, y; s, l) \xi_2(u, y) F_u(dy) \right\} \eta(ds, dl),$$

$\theta_1(u; s, l) = \theta_1(u; s, l) \mathbb{I}_{u > s}$  and  $\theta_2(u, y; s, l) = \theta_2(u, y; s, l) \mathbb{I}_{u > s}$ , which is quite interesting.

- 2) Furthermore, since a  $\mathbb{G}^{\tilde{\tau}, L}$ -stopping time might not be a  $\mathbb{G}$ -stopping time and the optional projection of a  $(P, \mathbb{G}^{\tilde{\tau}, L})$ -local martingale on  $\mathbb{G}$  might not be a  $(P, \mathbb{G})$ -local martingale, the proof of Proposition 3.3 in Callegaro, Jeanblanc and Zargari(2010) is not strict. We can also use the proof of Proposition 5.9 in El Karoui, Jeanblanc and Jiao(2009) to write  $X_t$  into the following form

$$X_t = X_1(t)\mathbb{I}_{t < \tilde{\tau}} + X_2(t; \tilde{\tau}, L)\mathbb{I}_{t \geq \tilde{\tau}}$$

where

$$\begin{aligned} X_1(t) &= m_t + \int_0^t \frac{1}{G_{u-}} \left\{ \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \left\{ \xi_1(u)' \theta_1(u; s, l) \right. \right. \\ &\quad \left. \left. + \int_E \theta_2(u, y; s, l) \xi_2(u, y) F_u(dy) \right\} \eta(ds, dl) \right\} du \\ X_2(t; s, l) &= \{m_t - X_1(s)\} \mathbb{I}_{t > s} - \int_s^t \left\{ \xi_1(u)' \theta_1(u; s, l) + \int_E \theta_2(u, y; s, l) \xi_2(u, y) F_u(dy) \right\} du, \end{aligned}$$

and show that  $\{X_2(t; s, l)p_t(s, l); t \geq s\}$  is a  $(P, \mathbb{F})$ -**local** martingale and  $\{X_1(t)G_t + \int_0^t \int_{\mathbb{R}} X_2(u; u, l)p_u(u, l)\eta(du, dl); t \geq 0\}$  is a  $(P, \mathbb{F})$ -**local** martingale. However, we can not use Theorem 3.1, since **the sequence of the localization stopping times of  $\{X_2(t; s, l)p_t(s, l); t \geq s\}$  depends on  $(s, l)$  which is uncountable infinite**, one can see that the proof of Proposition 5.9 in El Karoui-Jeanblanc-Jiao(2009) is not strict. Here we would like to provide a strict proof based on Proposition 4.5.

*Proof of Theorem 4.6.* Let  $m$  be a  $(P, \mathbb{F})$ -local martingale, then  $m$  is a  $(P^*, \mathbb{F})$ -local martingale which is also a  $(P^*, \mathbb{G}^{\tilde{\tau}, L})$ -local martingale. Since  $\mathbb{F} \subset \mathbb{G} \subset \mathbb{G}^{\tilde{\tau}, L}$ , thus  $m$  is a  $(P^*, \mathbb{G})$ -local martingale. Furthermore, since  $\frac{1}{p_t(\tilde{\tau}, L)}$  is the density process of  $P^*$  with respect to  $(P, \mathbb{G}^{\tilde{\tau}, L})$ , one can see that the density process of  $P^*$  with respect to  $(P, \mathbb{G})$  is given by

$$\begin{aligned} L_t^* &= E\left[\frac{1}{p_t(\tilde{\tau}, L)} \middle| \mathcal{G}_t\right] \\ &= \frac{1}{G_t} E\left(\int_t^\infty \int_{\mathbb{R}} \frac{1}{p_t(u, l)} p_t(u, l) \eta(du, dl) \middle| \mathcal{F}_s\right) \mathbb{I}_{t < \tilde{\tau}} + \frac{1}{p_t(\tilde{\tau}, L)} \mathbb{I}_{t \geq \tilde{\tau}} \\ &= \frac{1}{G_t} E\left(\int_t^\infty \int_{\mathbb{R}} \eta(du, dl) \middle| \mathcal{F}_s\right) \mathbb{I}_{t < \tilde{\tau}} + \frac{1}{p_t(\tilde{\tau}, L)} \mathbb{I}_{t \geq \tilde{\tau}} \\ &= \frac{G_t^*}{G_t} \mathbb{I}_{t < \tilde{\tau}} + \frac{1}{p_t(\tilde{\tau}, L)} \mathbb{I}_{t \geq \tilde{\tau}}, \end{aligned}$$

thus

$$\frac{1}{L_t^*} = \frac{G_t}{G_t^*} \mathbb{I}_{t < \tilde{\tau}} + p_t(\tilde{\tau}, L) \mathbb{I}_{t \geq \tilde{\tau}} = Z_t^*.$$

To prove  $X$  is a  $(P, \mathbb{G})$ -local martingale, we need only to prove that  $X_t Z_t^*$  is a  $(P^*, \mathbb{G})$ -local

martingale. As a matter of fact, one can see from Itô's formula that

$$\begin{aligned}
X_t Z_t^* &= m_0 + \int_0^t Z_{u-}^* dm_u + \int_0^t Z_{u-}^* \left\{ \frac{1}{G_{u-}} \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \left\{ \xi_1(u)' \theta_1(u; s, l) \right. \right. \\
&\quad \left. \left. + \int_E \theta_2(u, y; s, l) \xi_2(u, y) F_u(dy) \right\} \eta(ds, dl) \mathbb{I}_{u \leq \tilde{\tau}} \right. \\
&\quad \left. - \left\{ \xi_1(u)' \theta_1(u; \tilde{\tau}, L) + \int_E \theta_2(u, y; \tilde{\tau}, L) \xi_2(u, y) F_u(dy) \right\} \mathbb{I}_{u > \tilde{\tau}} \right\} du \\
&\quad + \int_0^t X_{u-} dZ_u^* + [X, Z^*]_t \\
&= m_0 + \int_0^t Z_{u-}^* dm_u + \int_0^t X_{u-} dZ_u^* \\
&\quad + \int_0^t Z_{u-}^* \left\{ \frac{1}{G_{u-}} \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \left\{ \xi_1(u)' \theta_1(u; s, l) \right. \right. \\
&\quad \left. \left. + \int_E \theta_2(u, y; s, l) \xi_2(u, y) F_u(dy) \right\} \eta(ds, dl) \mathbb{I}_{u \leq \tilde{\tau}} \right. \\
&\quad \left. - \left\{ \xi_1(u)' \theta_1(u; \tilde{\tau}, L) + \int_E \theta_2(u, y; \tilde{\tau}, L) \xi_2(u, y) F_u(dy) \right\} \mathbb{I}_{u > \tilde{\tau}} \right\} du \\
&\quad - \int_0^t Z_{u-}^* \left\{ \frac{1}{G_{u-}} \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_1(u; s, l) \eta(ds, dl) \mathbb{I}_{u \leq \tilde{\tau}} \right. \\
&\quad \left. - \theta_1(u; \tilde{\tau}, L) \mathbb{I}_{u > \tilde{\tau}} \right\} \xi_1(u) du \\
&\quad - \int_0^t \int_E Z_{u-}^* \left\{ \frac{1}{G_{u-}} \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_2(u, y; s, l) \eta(ds, dl) \mathbb{I}_{u \leq \tilde{\tau}} \right. \\
&\quad \left. - \theta_2(u, y; \tilde{\tau}, L) \mathbb{I}_{u > \tilde{\tau}} \right\} \xi_2(u, y) \mu(du, dy) \\
&= m_0 + \int_0^t Z_{u-}^* dm_u + \int_0^t X_{u-} dZ_u^* \\
&\quad - \int_0^t \int_E Z_{u-}^* \left\{ \frac{1}{G_{u-}} \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_2(u, y; s, l) \eta(ds, dl) \mathbb{I}_{u \leq \tilde{\tau}} \right. \\
&\quad \left. - \theta_2(u, y; \tilde{\tau}, L) \mathbb{I}_{u > \tilde{\tau}} \right\} \xi_2(u, y) \{ \mu(du, dy) - \nu(du, dy) \}.
\end{aligned}$$

Since both  $m$  and  $Z^*$  are  $(P^*, \mathbb{G})$ -local martingale, one can see that  $XZ^*$  is a  $(P^*, \mathbb{G})$ -local martingale, which completes the proof.  $\square$

**Corollary 4.7.** *Assume conditions of Theorem 4.6 hold, and let*

$$\begin{aligned}
W^{\mathbb{G}}(t) &:= W(t) + \int_0^t \left\{ \frac{1}{G_{u-}} \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_1(u; s, l) \eta(ds, dl) \mathbb{I}_{u \leq \tilde{\tau}} \right. \\
&\quad \left. - \theta_1(u; \tilde{\tau}, L) \mathbb{I}_{u > \tilde{\tau}} \right\} du,
\end{aligned}$$

then  $W^{\mathbb{G}}$  is a  $(P, \mathbb{G})$ -Brownian motion.

**Corollary 4.8.** *Assume conditions of Theorem 4.6 hold, and let*

$$\begin{aligned}
\nu^{\mathbb{G}}(du, dy) &:= \nu(du, dy) + \left\{ \frac{1}{G_{u-}} \int_0^u \int_{\mathbb{R}} p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_2(u, y; s, l) \eta(ds, dl) \mathbb{I}_{u \leq \tilde{\tau}} \right. \\
&\quad \left. - \theta_2(u, y; \tilde{\tau}, L) \mathbb{I}_{u > \tilde{\tau}} \right\} F_u(dy) du,
\end{aligned}$$



then  $\nu^{\mathbb{G}}(du, dy)$  is the compensator of  $\mu(du, dy)$  with respect to  $(P, \mathbb{G})$ .

**Remark.** From Theorem 4.6, we get the  $\mathbb{G}$ -decomposition of a  $\mathbb{F}$  martingale, the  $(P, \mathbb{G})$ -Brownian motion and the compensator of  $\mu(du, dy)$  with respect to  $(P, \mathbb{G})$  explicitly.

## 5 The representation of a $(P, \mathbb{G})$ -martingale

First, we have the following representation for a  $(P^*, \mathbb{G})$ -martingale:

**Theorem 5.1.** *Let  $M_t = M_1(t)\mathbb{I}_{t < \tilde{\tau}} + M_2(t; \tilde{\tau}, L)\mathbb{I}_{t \geq \tilde{\tau}}$  be a u.i.  $(P^*, \mathbb{G})$  martingale, then there exists a  $\mathbb{G}$ -predictable process  $\xi$  and a  $\widetilde{\mathcal{P}}(\mathbb{G})$ -measurable function  $\zeta(u, y)$  such that*

$$\begin{aligned} M_t &= M_0 + \int_0^t \xi(u)' dW(u) + \int_0^t \int_E \zeta(u, y) \{\mu(du, dy) - \nu(du, dy)\} \\ &\quad + M_2(\tilde{\tau}; \tilde{\tau}, L)\mathbb{I}_{t \geq \tilde{\tau}} - \int_0^{t \wedge \tilde{\tau}} \int_{\mathbb{R}} \frac{M_2(u; u, l)}{G_u^*} \eta(du, dl) \\ &\quad - M_1(\tilde{\tau}-)\mathbb{I}_{t \geq \tilde{\tau}} - \int_0^{t \wedge \tilde{\tau}} \frac{M_1(u-)}{G_u^*} dG_u^*, \end{aligned} \quad (5.1)$$

*Proof.* From Corollary 3.3, one can see that  $\{M_1(t)G_t^* + \int_0^t \int_{\mathbb{R}} M_2(u; u, l)\eta(du, dl); t \geq 0\}$  and  $\{M_2(t; s, l); t \geq s\}$  are  $(P^*, \mathbb{F})$ -martingales for  $\eta$ -almost every  $u \geq 0$  and  $l \in \mathbb{R}$ , thus there exist  $\mathbb{F}$ -predictable processes  $\xi_1$  and  $\xi_2(s, l)$  and  $\widetilde{\mathcal{P}}(\mathbb{F})$ -measurable functions  $\zeta_1(u, y)$  and  $\zeta_2(u, y; s, l)$  such that

$$\begin{aligned} M_1(t)G_t^* &+ \int_0^t \int_{\mathbb{R}} M_1(u; u, l)\eta(du, dl) \\ &= M_1(0) + \int_0^t \xi_1(u)' dW(u) + \int_0^t \int_E \zeta_1(u, y) \{\mu(du, dy) - \nu(du, dy)\} \text{ and} \\ M_2(t; s, l) &= M_2(s; s, l) + \int_s^t \xi_2(u; s, l)' dW(u) + \int_s^t \int_E \zeta_1(u, y; s, l) \{\mu(du, dy) - \nu(du, dy)\}. \end{aligned}$$

Thus

$$\begin{aligned} M_1(t)G_t^* &= M_1(0) + \int_0^t \xi_1(u)' dW(u) + \int_0^t \int_E \zeta_1(u, y) \{\mu(du, dy) - \nu(du, dy)\} \\ &\quad - \int_0^t \int_{\mathbb{R}} M_1(u; u, l)\eta(du, dl), \end{aligned}$$

one can see from Itô's formula that

$$\begin{aligned} M_1(t) &= M_1(t)G_t^* \frac{1}{G_t^*} \\ &= M_1(0) + \int_0^t \frac{\xi_1(u)'}{G_{u-}^*} dW(u) + \int_0^t \int_E \frac{\zeta_1(u, y)}{G_{u-}^*} \{\mu(du, dy) - \nu(du, dy)\} \\ &\quad - \int_0^t \int_{\mathbb{R}} \frac{M_1(u; u, l)}{G_{u-}^*} \eta(du, dl) + \int_0^t M_1(u-)G_{u-}^* d\left(\frac{1}{G_u^*}\right) \\ &= M_1(0) + \int_0^t \frac{\xi_1(u)'}{G_{u-}^*} dW(u) + \int_0^t \int_E \frac{\zeta_1(u, y)}{G_{u-}^*} \{\mu(du, dy) - \nu(du, dy)\} \\ &\quad - \int_0^t \int_{\mathbb{R}} \frac{M_1(u; u, l)}{G_{u-}^*} \eta(du, dl) - \int_0^t \frac{M_1(u-)}{G_u^*} dG_u^*. \end{aligned}$$

Thus

$$\begin{aligned}
M_t &= M_1(t)\mathbb{I}_{t < \tilde{\tau}} + M_2(t; \tilde{\tau}, L)\mathbb{I}_{t \geq \tilde{\tau}} \\
&= M_1(t \wedge \tilde{\tau}) - M_1(\tilde{\tau})\mathbb{I}_{t \geq \tilde{\tau}} + M_2(t; \tilde{\tau}, L)\mathbb{I}_{t \geq \tilde{\tau}} \\
&= M_1(0) + \int_0^{t \wedge \tilde{\tau}} \frac{\xi_1(u)'}{G_{u-}^*} dW(u) + \int_0^{t \wedge \tilde{\tau}} \int_E \frac{\zeta_1(u, y)}{G_{u-}^*} \{\mu(du, dy) - \nu(du, dy)\} \\
&\quad - \int_0^{t \wedge \tilde{\tau}} \int_{\mathbb{R}} \frac{M_1(u; u, l)}{G_{u-}^*} \eta(du, dl) - \int_0^{t \wedge \tilde{\tau}} \frac{M_1(u-)}{G_u^*} dG_u^* \\
&\quad - M_1(\tilde{\tau})\mathbb{I}_{t \geq \tilde{\tau}} + M_2(t; \tilde{\tau}, L)\mathbb{I}_{t \geq \tilde{\tau}} \\
&= M_0 + \int_0^t \xi(u)' dW(u) + \int_0^t \int_E \zeta(u, y) \{\mu(du, dy) - \nu(du, dy)\} \\
&\quad + M_2(\tilde{\tau}; \tilde{\tau}, L)\mathbb{I}_{t \geq \tilde{\tau}} - \int_0^{t \wedge \tilde{\tau}} \int_{\mathbb{R}} \frac{M_2(u; u, l)}{G_u^*} \eta(du, dl) \\
&\quad - M_1(\tilde{\tau})\mathbb{I}_{t \geq \tilde{\tau}} - \int_0^{t \wedge \tilde{\tau}} \frac{M_1(u-)}{G_u^*} dG_u^*,
\end{aligned}$$

where

$$\begin{aligned}
\xi(u) &= \frac{\xi_1(u)}{G_{u-}^*} \mathbb{I}_{u \leq \tilde{\tau}} + \xi_2(u; \tilde{\tau}, L) \mathbb{I}_{u > \tilde{\tau}}, \\
\zeta(u, y) &= \frac{\zeta_1(u, y)}{G_{u-}^*} \mathbb{I}_{u \leq \tilde{\tau}} + \zeta_2(u, y; \tilde{\tau}, L) \mathbb{I}_{u > \tilde{\tau}}.
\end{aligned}$$

Since the  $(P^*, \mathbb{F})$ -martingale  $\{M_1(t)G_t^* + \int_0^t \int_{\mathbb{R}} M_2(u; u, l)\eta(du, dl); t \geq 0\}$  has no jumps at  $\tilde{\tau}$  as a  $(P^*, \mathbb{G})$ -martingale and  $G^*$  is continuous, one can see that  $M_1(\tilde{\tau}) = M_1(\tilde{\tau}-)$ , a.s. and (5.1) follows.  $\square$

Now we turn to prove the predictable representation theorem for a  $(P, \mathbb{G})$ -martingale. Similar to Lemma 4.1, we have the following lemma

**Lemma 5.2.** *For any positive  $\mathcal{O}(\mathbb{F}) \times \mathcal{B}$ -measurable function  $f(s, l)$  such that  $E_P(|f(\tilde{\tau}, L)|) < \infty$ , let*

$$A_t^f := \int_0^t \int_{\mathbb{R}} \frac{f(s, l)}{G_{s-}} p_s(s, l) \eta(ds, dl),$$

then  $A^f$  is a continuous increasing process and

$$M_t^f = f(\tilde{\tau}, L)\mathbb{I}_{t \geq \tilde{\tau}} - A_{t \wedge \tilde{\tau}}^f$$

is a  $(P, \mathbb{G})$ -martingale, i.e.,  $A_{\cdot \wedge \tilde{\tau}}^f$  is the  $(P, \mathbb{G})$ -compensator of  $f(\tilde{\tau}, L)\mathbb{I}_{t \geq \tilde{\tau}}$ .

*Proof.* Similar to the proof of Lemma 4.1, one can see that for any  $t_1 < t_2$ ,

$$\begin{aligned}
E[f(\tilde{\tau}, L)\mathbb{I}_{t_1 < \tilde{\tau} \leq t_2} | \mathcal{F}_{t_1}] &= \int_{t_1}^{t_2} \int_{\mathbb{R}} f(s, l) p_{t_1}(s, l) \eta(ds, dl) \\
&= E\left[\int_{t_1}^{t_2} \int_{\mathbb{R}} f(s, l) p_s(s, l) \eta(ds, dl) \middle| \mathcal{F}_{t_1}\right]
\end{aligned}$$

and that

$$\begin{aligned}
&E\left[\int_{t_1}^{t_2} \int_{\mathbb{R}} \frac{\mathbb{I}_{s \leq \tilde{\tau}}}{G_{s-}} p_s(s, l) f(s, l) \eta(ds, dl) \middle| \mathcal{F}_{t_1}\right] \\
&= E\left[\int_{t_1}^{t_2} \int_{\mathbb{R}} \frac{E[\mathbb{I}_{s \leq \tilde{\tau}} | \mathcal{F}_s]}{G_{s-}} p_s(s, l) f(s, l) \eta(ds, dl) \middle| \mathcal{F}_{t_1}\right] \\
&= \int_{t_1}^{t_2} \int_{\mathbb{R}} f(s, l) p_s(s, l) \eta(ds, dl),
\end{aligned}$$

and the rest is completely the same as the proof of Lemma 4.1.  $\square$

**Theorem 5.3.** *Let  $M_t := M_1(t)\mathbb{I}_{t < \tilde{\tau}} + M_2(t; \tilde{\tau}, L)\mathbb{I}_{t \geq \tilde{\tau}}$  be a u.i.  $(P, \mathbb{G})$ -martingale, then there exists a  $\mathbb{F}$ -predictable process  $\xi$  and a  $\mathcal{P}(\mathbb{F})$ -measurable function  $\zeta(u, y)$  such that*

$$\begin{aligned} M_t &= M_1(0) + \int_0^t \xi(u)' dW^{\mathbb{G}}(u) + \int_0^t \int_E \zeta(u, y) \{ \mu(du, dy) - \nu^{\mathbb{G}}(du, dy) \} \\ &\quad + M_2(\tilde{\tau}; \tilde{\tau}, L)\mathbb{I}_{t \geq \tilde{\tau}} - \int_0^{t \wedge \tilde{\tau}} \int_R \frac{M_2(u; u, l)}{G_{u-}} p_u(u, l) \eta(du, dl) \\ &\quad - M_1(\tilde{\tau}-)\mathbb{I}_{t \geq \tilde{\tau}} - \int_0^{t \wedge \tilde{\tau}} \int_R \frac{M_1(u-)}{G_{u-}} p_s(s, l) \eta(ds, dl). \end{aligned} \quad (5.2)$$

**Remark.** By Lemma 5.2, one can see that  $M_2(\tilde{\tau}; \tilde{\tau}, L) - \int_0^{t \wedge \tilde{\tau}} \int_R \frac{M_2(u; u, l)}{G_{u-}} p_u(u, l) \eta(du, dl)$

and  $M_1(\tilde{\tau}-)\mathbb{I}_{t \geq \tilde{\tau}} + \int_0^{t \wedge \tilde{\tau}} \int_R \frac{M_1(u-)}{G_{u-}} p_s(s, l) \eta(ds, dl)$  are  $(P, \mathbb{G})$ -martingales. With the non-trivial conditional density  $p_t(\tilde{\tau}, L)$ , Theorem 5.3 may be viewed as the more general result of Theorem 5.2 and the representation of a  $(P, \mathbb{G})$ -martingale is different from Callegaro, Jeanblanc and Zargari(2010)<sup>[5]</sup>.

*Proof of Theorem 5.3.* From Theorem 3.1, one can see that  $\{M_2(t; u, l)p_t(u, l)(t \geq u)\}$  and  $\{M_1(t)G_t + \int_0^t \int_R M_2(u; u, l)p_u(u, l)\eta(du, dl)(t \geq 0)\}$  are  $(P, \mathbb{F})$ -martingales for  $\eta$ -almost every  $u \geq 0$  and  $l \in \mathbb{R}$ , thus there exists  $\mathbb{F}$ -predictable processes  $\xi_1(u)$ ,  $\xi_2(u; s, l)$  and  $\mathcal{P}(\mathbb{F})$ -measurable functions  $\zeta_1(u, y)$ ,  $\zeta_2(u, y; s, l)$ , such that

$$\begin{aligned} M_1(t)G_t + \int_0^t \int_R M_2(u; u, l)p_t(u, l)\eta(du, dl) \\ &= M_1(0) + \int_0^t \xi_1(u) dW(u) + \int_0^t \int_E \zeta_1(u, y) \{ \mu(du, dy) - \nu(du, dy) \} \text{ and} \\ M_2(t; s, l)p_t(s, l) &= M_2(s; s, l)p_s(s, l) + \int_s^t \xi_2(u; s, l)' dW(u) \\ &\quad + \int_s^t \int_E \zeta_2(u, y; s, l) \{ \mu(du, dy) - \nu(du, dy) \}. \end{aligned}$$

Let  $A_u^{\mathbb{G}} := -\frac{1}{G_{u-}} \int_0^u \int_R p_s(s, l) \eta(ds, dl)$ ,  $\alpha_1(u) := \frac{1}{G_{u-}} \int_0^u \int_R p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_1(u; s, l) \eta(ds, dl)$ ,  $\alpha_2(u, y) := \frac{1}{G_{u-}} \int_0^u \int_R p_s(s, l) Z_{s,l}^{\theta_1, \theta_2}(u-) \theta_2(u; s, l) \eta(ds, dl)$ , then one can see that  $G_t$  can be rewritten into the following form

$$G_t = 1 - \int_0^t G_{u-} dA_u^{\mathbb{G}} - \int_0^t G_{u-} \alpha_1(u)' dW(u) - \int_0^t \int_E G_{u-} \alpha_2(u, y) \{ \mu(du, dy) - \nu(du, dy) \},$$

which implies that

$$\begin{aligned} \frac{1}{G_t} &= 1 + \int_0^t \frac{1}{G_{u-}} dA_u^{\mathbb{G}} + \int_0^t \frac{1}{G_{u-}} \alpha_1(u)' dW(u) - \int_0^t \int_E \frac{1}{G_{u-}} \alpha_2(u, y) \nu(du, dy) \\ &\quad + \int_0^t \frac{1}{G_{u-}} \| \alpha_1(u) \|^2 du + \int_0^t \int_E \frac{1}{G_{u-}} \left\{ \frac{1}{1 - \alpha_2(u, y)} - 1 \right\} \mu(du, dy). \end{aligned}$$

Since

$$\begin{aligned} p_t(\tilde{\tau}, L) &= p_{\tilde{\tau}}(\tilde{\tau}, L)\mathbb{I}_{t \geq \tilde{\tau}} + \int_0^t p_{u-}(\tilde{\tau}, L)\theta_1(u; \tilde{\tau}, L)' \mathbb{I}_{u > \tilde{\tau}} dW(u) \\ &\quad + \int_0^t \int_E p_{u-}(\tilde{\tau}, L)\theta_2(u, y; \tilde{\tau}, L)\mathbb{I}_{u > \tilde{\tau}} \{\mu(du, dy) - \nu(du, dy)\}, \end{aligned}$$

thus

$$\begin{aligned} \frac{1}{p_t(\tilde{\tau}, L)} &= \frac{1}{p_{\tilde{\tau}}(\tilde{\tau}, L)}\mathbb{I}_{t \geq \tilde{\tau}} - \int_0^t \frac{1}{p_{u-}(\tilde{\tau}, L)}\theta_1(u; \tilde{\tau}, L)' \mathbb{I}_{u > \tilde{\tau}} dW(u) \\ &\quad + \int_0^t \int_E \frac{1}{p_{u-}(\tilde{\tau}, L)}\theta_2(u, y; \tilde{\tau}, L)\mathbb{I}_{u > \tilde{\tau}} \nu(du, dy) \\ &\quad + \int_0^t \frac{1}{p_{u-}(\tilde{\tau}, L)}\|\theta_1(u; \tilde{\tau}, L)\|^2 \mathbb{I}_{u > \tilde{\tau}} du \\ &\quad + \int_0^t \int_E \frac{1}{p_{u-}(\tilde{\tau}, L)} \left\{ \frac{1}{1 + \theta_2(u, y; \tilde{\tau}, L)} - 1 \right\} \mathbb{I}_{u > \tilde{\tau}} \mu(du, dy). \end{aligned}$$

From Itô's formula, one has

$$\begin{aligned} M_1(t) &= M_1(t)G_t \frac{1}{G_t} \\ &= M_1(0) + \int_0^t \frac{\xi_1(u)'}{G_{u-}} dW(u) + \int_0^t \int_E \frac{\zeta_1(u, y)}{G_{u-}} \{\mu(du, dy) - \nu(du, dy)\} \\ &\quad - \int_0^t \int_R \frac{M_2(u; u, l)}{G_{u-}} p_u(u, l) \eta(du, dl) + \int_0^t M_1(u-) G_{u-} d\left(\frac{1}{G_u}\right) + \int_0^t d[M_1 G, \frac{1}{G}]_u \\ &= M_1(0) + \int_0^t \frac{\xi_1(u)'}{G_{u-}} dW(u) + \int_0^t \int_E \frac{\zeta_1(u, y)}{G_{u-}} \{\mu(du, dy) - \nu(du, dy)\} \\ &\quad - \int_0^t \int_R \frac{M_2(u; u, l)}{G_{u-}} p_u(u, l) \eta(du, dl) \\ &\quad + \int_0^t M_1(u-) dA_u^G + \int_0^t M_1(u-) \alpha_1(u)' dW(u) - \int_0^t \int_E M_1(u-) \alpha_2(u, y) \nu(du, dy) \\ &\quad + \int_0^t M_1(u-) \|\alpha_1(u)\|^2 du + \int_0^t \int_E M_1(u-) \left\{ \frac{1}{1 - \alpha_2(u, y)} - 1 \right\} \mu(du, dy) \\ &\quad + \int_0^t \frac{1}{G_{u-}} \xi_1(u)' \alpha_1(u) du + \int_0^t \int_E \frac{1}{G_{u-}} \frac{\zeta_1(u, y) \alpha_2(u, y)}{1 - \alpha_2(u, y)} \mu(du, dy), \end{aligned}$$

By calculation, we get

$$\begin{aligned} M_1(t \wedge \tilde{\tau}) &= M_1(0) + \int_0^{t \wedge \tilde{\tau}} M_1(u-) dA_u^G - \int_0^{t \wedge \tilde{\tau}} \int_R \frac{M_2(u; u, l)}{G_{u-}} p_u(u, l) \eta(du, dl) \\ &\quad + \int_0^{t \wedge \tilde{\tau}} \left\{ \frac{\xi_1(u)}{G_{u-}} + M_1(u-) \alpha_1(u) \right\}' dW^G(u) \\ &\quad + \int_0^{t \wedge \tilde{\tau}} \int_E \frac{1}{1 - \alpha_2(u, y)} \left\{ \frac{\zeta_1(u, y)}{G_{u-}} + M_1(u-) \alpha_2(u, y) \right\} \{\mu(du, dy) - \nu^G(du, dy)\} \end{aligned}$$

here we have used the following equalities

$$W^G(t) = W(t) + \int_0^{\tilde{\tau}} \alpha_1(u) du - \int_{\tilde{\tau}}^t \theta_1(u; \tilde{\tau}, L) du,$$

$$\nu^{\mathbb{G}}(du, dy) = \nu(du, dy) + \alpha_2(u, y)\nu(du, dy)\mathbb{I}_{u \leq \tilde{\tau}} - \theta_2(u, y; \tilde{\tau}, L)\nu(du, dy)\mathbb{I}_{u > \tilde{\tau}}.$$

By Itô's formula again, one can see that

$$\begin{aligned} M_2(t; \tilde{\tau}, L) &= M_2(t; \tilde{\tau}, L)p_t(\tilde{\tau}, L)\frac{1}{p_t(\tilde{\tau}, L)} \\ &= M_2(\tilde{\tau}; \tilde{\tau}, L) + \int_{\tilde{\tau}}^t \frac{\xi_2(u; \tilde{\tau}, L)'}{p_{u-}(\tilde{\tau}, L)}dW(u) + \int_{\tilde{\tau}}^t \int_E \frac{\zeta_2(u, y; \tilde{\tau}, L)}{p_{u-}(\tilde{\tau}, L)}\{\mu(du, dy) - \nu(du, dy)\} \\ &\quad + \int_{\tilde{\tau}}^t M_2(u-; \tilde{\tau}, L)p_{u-}(\tilde{\tau}, L)d\left(\frac{1}{p_u(\tilde{\tau}, L)}\right) + \int_{\tilde{\tau}}^t d[M_2(\cdot; \tilde{\tau}, L)p(\tilde{\tau}, L), \frac{1}{p(\tilde{\tau}, L)}]_u \\ &= M_2(\tilde{\tau}; \tilde{\tau}, L) + \int_{\tilde{\tau}}^t \left\{ \frac{\xi_2(u; \tilde{\tau}, L)'}{p_{u-}(\tilde{\tau}, L)} - M_2(u-; \tilde{\tau}, L)\theta_1(u; \tilde{\tau}, L) \right\} dW^{\mathbb{G}}(u) \\ &\quad + \int_{\tilde{\tau}}^t \int_E \frac{1}{1 + \theta_2(u, y; \tilde{\tau}, L)} \left\{ \frac{\zeta_2(u, y; \tilde{\tau}, L)}{p_{u-}(\tilde{\tau}, L)} - M_2(u-; \tilde{\tau}, L)\theta_2(u, y; \tilde{\tau}, L) \right\} \\ &\quad \{ \mu(du, dy) - \nu^{\mathbb{G}}(du, dy) \}. \end{aligned}$$

Thus

$$\begin{aligned} M_t &= M_1(t)\mathbb{I}_{t < \tilde{\tau}} + M_2(t; \tilde{\tau}, L)\mathbb{I}_{t \geq \tilde{\tau}} \\ &= M_1(t \wedge \tilde{\tau}) - M_1(\tilde{\tau})\mathbb{I}_{t \geq \tilde{\tau}} + M_2(t; \tilde{\tau}, L)\mathbb{I}_{t \geq \tilde{\tau}} \\ &= M_1(0) + \int_0^t \xi(u)'dW^{\mathbb{G}}(u) + \int_0^t \int_E \zeta(u, y)\{\mu(du, dy) - \nu^{\mathbb{G}}(du, dy)\} \\ &\quad + M_2(\tilde{\tau}; \tilde{\tau}, L) - \int_0^{t \wedge \tilde{\tau}} \int_R \frac{M_2(u; u, l)}{G_{u-}} p_u(u, l)\eta(du, dl) \\ &\quad - M_1(\tilde{\tau})\mathbb{I}_{t \geq \tilde{\tau}} - \int_0^{t \wedge \tilde{\tau}} \int_R \frac{M_1(u-)}{G_{u-}} p_s(s, l)\eta(ds, dl), \end{aligned}$$

where

$$\begin{aligned} \xi(u) &= \left\{ \frac{\xi_1(u)}{G_{u-}} + M_1(u-)\alpha_1(u) \right\} + \left\{ \frac{\xi_2(u; \tilde{\tau}, L)'}{p_{u-}(\tilde{\tau}, L)} - M_2(u-; \tilde{\tau}, L)\theta_1(u; \tilde{\tau}, L) \right\}, \\ \zeta(u, y) &= \frac{1}{1 - \alpha_2(u, y)} \left\{ \frac{\zeta_1(u, y)}{G_{u-}} + M_1(u-)\alpha_2(u, y) \right\} \\ &\quad + \frac{1}{1 + \theta_2(u, y; \tilde{\tau}, L)} \left\{ \frac{\zeta_2(u, y; \tilde{\tau}, L)}{p_{u-}(\tilde{\tau}, L)} - M_2(u-; \tilde{\tau}, L)\theta_2(u, y; \tilde{\tau}, L) \right\}. \end{aligned}$$

Since both  $\left\{ M_1(t)G_t + \int_0^t \int_R M_2(u; u, l)p_u(u, l)\eta(du, dl)(t \geq 0) \right\}$  and  $G$  have no jump at  $\tilde{\tau}$  as the  $(P, \mathbb{G})$  semimartingales, one can see that  $G_{\tilde{\tau}} = G_{\tilde{\tau}-}$ , a.s. and  $M_1(\tilde{\tau})G_{\tilde{\tau}} = M_1(\tilde{\tau}-)G_{\tilde{\tau}-}$ , a.s., thus  $M_1(\tilde{\tau}) = M_1(\tilde{\tau}-)$ , a.s., which completes the proof.  $\square$

## 6 Conclusion

In this paper, we mainly study a new kind of progressive enlargement filtration. We deeply characterize the conditional density process and give the Doob-Meyer's decomposition of the survival process. We also discuss the necessary and sufficient conditions for a  $\mathbb{G}$ -martingale. By Lemma 4.1, we explicitly describe the  $\mathbb{G}$ -decomposition of a  $(P, \mathbb{F})$ -martingale and prove the martingale representation theorems which extend the traditional results.

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